山形大学紀要（工学）第12巻 第1号 昭和47年1月
Bull. of Yamagata Univ. Eng., Vol. 12 No.1 Jan. 1972

Analysis of Characteristics of D-C Servomotor with Large Rate Feedback Considering Friction Varying with Angular Position

Masayuki HATTORI,* Ryoichi TAKANASHI**
and Tsunetomo ANNO*

*Department of Electronic Engineering, Faculty of Engineering
**Department of Computer Engineering, Technical Junior College

Abstract

The characteristics of a servomotor with large rate feedback considering the friction torque varying with angular position are analysed. In analysis, the equation of the motor is approximated with a first order nonlinear differential equation. By the use of the analytical results, calculations are done for a typical d-c servomotor on the market, and the calculated results are compared with the experimental and analog simulated results.

As a result, it is clarified that the analytical results can express the characteristics of a d-c servomotor with large rate feedback quantitatively in small angular velocity, for instance, the negative slope phenomena in the characteristics of time-averaged angular velocity vs. time-averaged friction torque of the motor, which have been explained only qualitatively.

1 Introduction

In the past, friction torque of a d-c servomotor was considered to be a function of only angular velocity of the motor, while a friction torque characteristic in the range of lower speed was not clarified because of difficulties in measurement.

Recently the authors have measured the friction torque characteristic of a d-c servomotor in small angular velocity by applying rate feedback to the motor, and have clarified that the friction torque characteristic of a d-c servomotor is a function of not only angular velocity but also angular position of the motor.[1] Moreover, the authors have made simulation of a d-c servomotor with analog computer regarding the friction torque as the sum of friction torque components varying with angular velocity and angular position, and have obtained good results.[2]

However, if the friction torque varying with angular position is considered, the equation of a d-c servomotor becomes a second-order nonlinear differential equation and its analytical solution can not be obtained generally, and therefore analytical investigation of the motor has not been made.

In this paper, the characteristics of a d-c servomotor is analysed by neglecting the acceleration torque term by reason that, in equation of the motor, the acceleration torque term (second-order term) is very small compared with viscous damping term (first-order term) if rate feedback to the motor is sufficently large. And analysis is made in two cases in which the friction torque varying with angular position is assumed to be either sinusoidal or triangular.

Furthermore, a brief discussion on the effect of the acceleration torque term on the characteristics of the motor which is neglected in analysis is given.

As a result, it is clarified that the characteristics of a d-c servomotor with large rate feedback in small angular velocity can be investigated analytically.

2 Analysis of D-C Servomotor

2.1 Equation of D-C Servomotor with Rate Feedback

The equations of an armature-controlled d-c servomotor with rate feedback are assumed to be as follows, and the block diagram is shown in Fig. 1.

$$(v_i - \mathrm{K}_B\omega)\,\mathrm{K}_A = v_a \tag{1}$$

$$v_a = \mathrm{R}_a\, i_a + \mathrm{K}_b\omega \tag{2}$$

$$\mathrm{K}_t\, i_a = \mathrm{J}_m \frac{d\omega}{dt} + f \tag{3}$$

$$\left.\begin{aligned} f &= f_1(\omega) + f_2(\theta) \\ f_1(\omega) &= \mathrm{F}_L\,\omega + \mathrm{F}_C \frac{\omega}{|\omega|} \end{aligned}\right\} \tag{4}$$

where,

v_i	Input voltage,
K_B	Coefficent of rate feedback,
K_A	Amplifier gain,
v_a	Armature voltage,
i_a	Armature current,

R_a	Armature resistance,
K_b	Back emf constant,
K_t	Torque constant,
θ	Angular displacement,
ω	Angular velocity,
f	Friction torque,
$f_1(\omega)$	Friction torque component varying with ω,
$f_2(\theta)$	Friction torque component varying with θ,
F_C	Coulomb friction,
F_L	Viscous friction coefficent,
J_m	Moment of inertia.

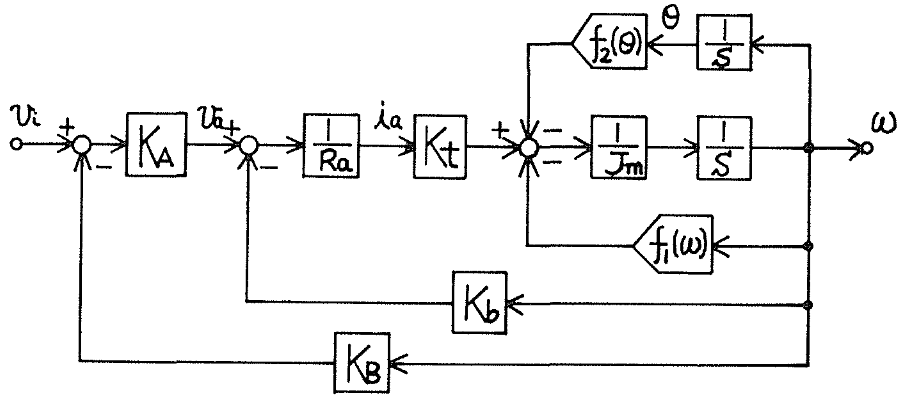

Fig. 1 Block diagram of a d-c servomotor with rate feedback

From Eqs. (1), (2), (3) and (4), by eliminating i_a and v_a, we obtain

$$J_m \frac{d^2\theta}{dt^2} + (P + F_L)\frac{d\theta}{dt} + F_C\frac{\omega}{|\omega|} + f_2(\theta) - \frac{K_t K_A}{R_a} v_i = 0 \tag{5}$$

where

$$P = \frac{K_t K_b + K_t K_A K_B}{R_a} \tag{6}$$

In Eq. (5), as P and F_L have the same dimension, the physical meaning of P can be regarded as the viscous friction coefficient which can be controlled externally. Therefore, as P increases, the variation of angular velocity of the motor due to the variation of the friction torque decreases. In Ref. (1), dimensionless quantity Γ is used to define the magnitude of rate feedback, where $\Gamma = 1 + K_A K_B / K_b$, and the relation P and Γ becomes

$$P = \frac{K_t K_b}{R_a}\Gamma. \tag{7}$$

In Eq. (6), when rate feedback is sufficiently large (or P is sufficiently large), since the effect of the acceleration torque term is very small compared with effect of the viscous damping term, it is assumed that the behavior of the motor can be investigated without considering the effect of the acceleration torque.

Therefore Eq. (5) becomes,

$$(P + F_L)\frac{d\theta}{dt} + F_C\frac{\omega}{|\omega|} + f_2(\theta) - \frac{K_t K_A}{R_a} v_i = 0. \tag{8}$$

In the following sections, we investigate Eq. (8) in two cases where $f_2(\theta)$ is either sinusoidal or triangular.

2.2 In Case $f_2(\theta)$ is a Sinusoidal Wave.

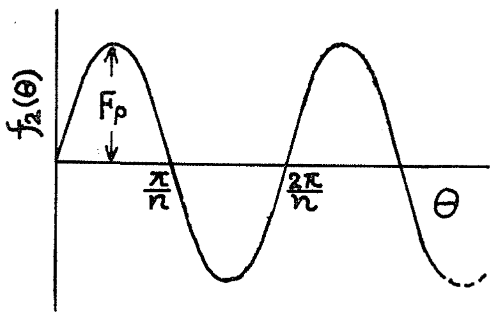

Fig. 2 The shape of $f_2(\theta)$, (sinusoidal wave)

Fig. 2 shows $f_2(\theta)$ which is assumed to be a sinusoidal wave.

In Fig. 2, F_P is the amplitude of the varying friction torque and n is a positive integer depending upon the construction of the motor.

We consider the case under the condition that v_i is a step function.

If the inequality

$$\frac{K_t K_A v_i}{R_a} > F_P + F_C$$

holds, then $\omega > 0$ and $F_C\,\omega/|\omega| = F_C$.

Therefore Eq. (8) becomes

$$(P + F_L)\frac{d\theta}{dt} + F_C + F_P \sin n\theta - \frac{K_t K_A v_i}{R_a} = 0. \tag{9}$$

For convenience, let

$$\left.\begin{aligned} \frac{K_t K_A v_i/R_a - F_C}{P + F_L} &= A \\ \frac{F_P}{P + F_L} &= B. \end{aligned}\right\} \tag{10}$$

Eq. (9) can be written as follows.

$$\frac{d\theta}{dt} = A - B\sin n\theta \tag{11}$$

By solving Eq. (11) for θ under the condition of $\theta=0$ when t=0, we obtain

$$\theta(t) = \frac{2}{n}\tan^{-1}\left\{\frac{\sqrt{A^2-B^2}}{A}\tan\left(\frac{n\sqrt{A^2-B^2}}{2}t-\varphi_0\right)+\frac{B}{A}\right\} \tag{12}$$

where

$$\varphi_0 = \sin^{-1}\frac{B}{A}.$$

Angular velocity of the motor can be obtained by differentiating $\theta(t)$ with respect to t.

$$\omega(t) = \frac{d\theta(t)}{dt} = \frac{A^2 - B^2}{A + B\sin(n\sqrt{A^2-B^2}\,t-\varphi_0)} \tag{13}$$

From Eq. (13), the following relations are obtained.

$\omega(0) = A$ when $t = 0$,

minimum value of $\omega(t) = A-B$ when $\sin(n\sqrt{A^2-B^2}\,t-\varphi_0) = 1$,

maximum value of $\omega(t) = A+B$ when $\sin(n\sqrt{A^2-B^2}\,t-\varphi_0) = -1$.

Using the relations mentioned above we can draw the shape of $\omega(t)$, and it is shown in Fig. 3.

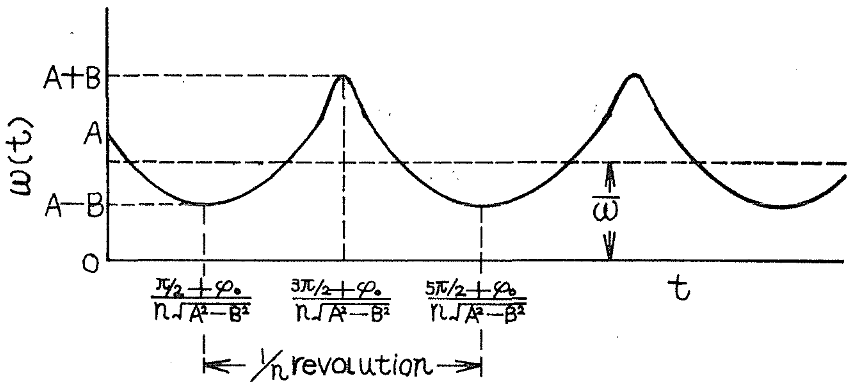

Fig. 3 The schematic view of $\omega(t)$ when $f_2(\theta)$ is sinusoidal wave

In Fig. 3, since the time required for one-nth revolution of the motor is $2\pi/n\sqrt{A^2-B^2}$, the time average of angular velocity: $\bar{\omega}$ is written as follows.

$$\bar{\omega} = \sqrt{A^2-B^2} \tag{14}$$

From Eq. (14), $\bar{\omega}$ can be also regarded as the geometric average of the maximum value and the minimum value of $\omega(t)$.

By the use of Eqs. (13) and (14), let us now describe $\omega(t)$ and angular acceleration : $\alpha(t)$ as the functions of $\bar{\omega}$ and B. Then, they are written as follows.

$$\omega(t) = \frac{\bar{\omega}^2}{\sqrt{\bar{\omega}^2+B^2} + B\sin(n\bar{\omega}t - \sin^{-1}B/\sqrt{\bar{\omega}^2+B^2})}, \tag{15}$$

$$\alpha(t) = \frac{d\omega(t)}{dt} = \frac{-B\bar{\omega}^3 n\cos(n\bar{\omega}t - \sin^{-1}B/\sqrt{\bar{\omega}^2+B^2})}{\left\{\sqrt{\bar{\omega}^2+B^2} + B\sin(n\bar{\omega}t - \sin^{-1}B/\sqrt{\bar{\omega}^2+B^2})\right\}}. \tag{16}$$

Let α_P be the maximum value of $\alpha(t)$ then it becomes

$$|\alpha_P| = \frac{\sqrt{2}\,n\bar{\omega}^3\sqrt{\sqrt{(\bar{\omega}^2+B^2)(\bar{\omega}^2+9B^2)} - (3B^2+\bar{\omega}^2)}}{5\bar{\omega}^2+9B^2-3\sqrt{(\bar{\omega}^2+B^2)(\bar{\omega}^2+9B^2)}}. \tag{17}$$

Friction torque varying with t : $f(t)$ can be obtained by multiplying armature current by torque constant. Therefore, from Eqs. (4), (8), and (10), $f(t)$ can be obtained as follows.

$$\begin{aligned} f(t) &= K_t i_a = \frac{K_t K_A}{R_a} v_i - P\omega(t) \\ &= (P+F_L)A + F_C - P\omega(t). \end{aligned} \tag{18}$$

Furthermore, applying Eqs. (10) and (14) to Eq. (18), we have

$$f(t) = \sqrt{(P+F_L)^2\bar{\omega}^2 + F_P^2} + F_C - P\omega(t). \tag{19}$$

Therefore the relation between $\bar{\omega}$ and the time average of friction torque: $\bar{f}$ ($\bar{\omega}-\bar{f}$ characteristic) can be obtained as follows by averaging both sides of Eq. (19) with respect to time.

$$\bar{f} = \sqrt{(P+F_L)^2\bar{\omega}^2 + F_P^2} + F_C - P\bar{\omega} \tag{20}$$

2.3 In Case $f_2(\theta)$ is a Triangular Wave

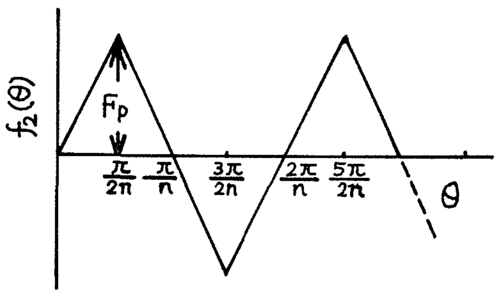

Fig. 4 The shape of $f_2(\theta)$, (triangular wave)

Fig. 4 shows $f_2(\theta)$ which is assumed to be a triangular wave. In this case, the behavior of the motor can be investigated by connecting the solutions of the partially linear differential equations. In the same manner as the last section, we assume the following inequality, i. e.,

$$\frac{K_t K_A v_i}{R_a} > F_P + F_C$$

therefore, $\omega > 0$.

From Fig. 4 and Eqs. (8) and (10), we have

$$\frac{d\theta}{dt} = A - \frac{2nB}{\pi}\theta \qquad \left(0 < \theta < \frac{\pi}{2n}\right), \tag{i}$$

$$\frac{d\theta}{dt} = A + \frac{2nB}{\pi}\theta - 2B \qquad \left(\frac{\pi}{2n} < \theta < \frac{3\pi}{2n}\right), \tag{ii}$$

$$\frac{d\theta}{dt} = A - \frac{2nB}{\pi}\theta + 4B \qquad \left(\frac{3\pi}{2n} < \theta < \frac{5\pi}{2n}\right), \tag{iii}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Solving Eq. (i) under the condition $\theta = 0$ when $t = 0$, we have

$$\left.\begin{aligned} \theta(t) &= \frac{A\pi}{2nB}\left(1 - e^{-\frac{2nB}{\pi}t}\right) \\ \omega(t) &= \frac{d\theta(t)}{dt} = Ae^{-\frac{2nB}{\pi}t} \\ \alpha(t) &= \frac{d\omega(t)}{dt} = -\frac{2nAB}{\pi}e^{-\frac{2nB}{\pi}t} \end{aligned}\right\} \tag{21}$$

Solving Eq. (ii) under the condition $\theta = \frac{\pi}{2n}$ when $t = 0$, we have

$$\left.\begin{aligned} \theta(t) &= -\frac{\pi(A-2B)}{2nB} + \frac{\pi(A-B)}{2nB} e^{\frac{2nB}{\pi}t} \\ \omega(t) &= (A-B)\, e^{\frac{2nB}{\pi}t} \\ \alpha(t) &= \frac{2nB(A-B)}{\pi} e^{\frac{2nB}{\pi}t} \end{aligned}\right\} \tag{22}$$

Solving Eq. (iii) under the condition $\theta = \frac{3\pi}{2n}$ when $t = 0$, we have

$$\left.\begin{aligned} \theta(t) &= \frac{\pi(A+4B)}{2nB} - \frac{\pi(A+B)}{2nB} e^{-\frac{2nB}{\pi}t} \\ \omega(t) &= (A+B)\, e^{-\frac{2nB}{\pi}t} \\ \alpha(t) &= -\frac{2nB(A+B)}{\pi} e^{-\frac{2nB}{\pi}t} \end{aligned}\right\} \tag{23}$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

Connecting Eqs. (21), (22) and (23), we can draw the shape of $\omega(t)$, and it is shown in Fig. 5.

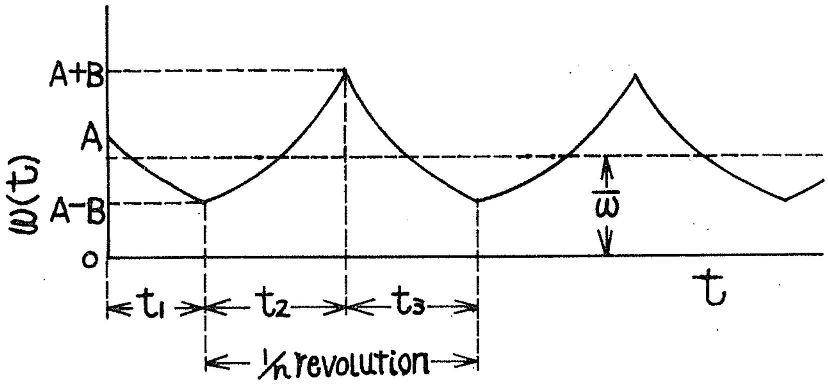

Fig. 5 The schematic view of $\omega(t)$ when $f_2(\theta)$ is triangular wave

In Fig. 5, t_1, t_2 and t_3 are determined from Eq. (21), (22) and (23) as follws.

$$t_1 = \frac{\pi}{2nB} \log \frac{A}{A-B}$$

$$t_2 = t_3 = \frac{\pi}{2nB} \log \frac{A+B}{A-B}$$

By the use of relations mentioned above the time average of $\omega(t)$ becomes

$$\bar{\omega} = \frac{2\pi}{n(t_2+t_3)} = \frac{2B}{\log \dfrac{A+B}{A-B}} . \tag{24}$$

Solving Eq. (24) for A, we have

$$A = \frac{1+e^{-\frac{2B}{\bar{\omega}}}}{1-e^{-\frac{2B}{\bar{\omega}}}} B . \tag{25}$$

Therefore, $\omega(t)$ and $\alpha(t)$ can be described as the functions of $\bar{\omega}$ and B by substituting Eq. (25) into Eqs. (21), (22) and (23).

The maximum value of angular acceleration is obtained by the use of Eqs. (22) or (23).

$$|\alpha_P| = \frac{2nB(A+B)}{\pi} = \frac{4nB^2}{\pi(1-e^{-\frac{2B}{\bar{\omega}}})} \tag{26}$$

Friction torque is obtained from Eq. (18) in the same way as the last section, and Fq. (18) is rewritten here.

$$f(t) = (P+F_L)A+F_C-P\omega(t) \tag{18}$$

Substituting Eq. (25) into Eq. (18), and using Eq. (10), we obtain

$$f(t) = \frac{1+e^{-\frac{2F_P}{(P+F_L)\bar{\omega}}}}{1-e^{-\frac{2F_P}{(P+F_L)\bar{\omega}}}} F_P + F_C - P\omega(t) . \tag{27}$$

Therefore, the relation between $\bar{\omega}$ and $\bar{f}$ is obtained by averaging both sides of Eq. (27) with respect to time

$$\bar{f} = \frac{1+e^{-\frac{2F_P}{(P+F_L)\bar{\omega}}}}{1-e^{-\frac{2F_P}{(P+F_L)\bar{\omega}}}} F_P + F_C - P\bar{\omega} \tag{28}$$

3 Theoretical Results and Discussion

3.1 Numerical Examples

In order to examine the results of analysis obtained in the last chapter, calculations are done for a d-c servomotor obtained from T Co.; (Rated Output 8 W, Rated Armature Voltage 26.5 V, No-Load Speed 15,000 r.p.m.). An analog simulation of this motor has been made in which $f_2(\theta)$ is approximated by the triangular wave. The numerical values used for calculations are the same data used for analog simulation, and they are,

$F_P = 68[\text{g-cm}]$, $R_a = 6.44[\Omega]$,

$F_C = 50[\text{g-cm}]$, $F_L = 0.54[\text{g-cm/rad/s}]$ $(\omega<50 \text{ rad/s})$,

$n = 14$.

As examples of the results, with respect to the shapes of $\omega(t)$ for $\bar{\omega} = 2$ rad/s and P=71.2 (or Γ=200), both measured and analog simulated results are shown in Fig. 6 and calculated results are shown in Fig. 7.

With respect to $\bar{\omega}$ - $\bar{f}$ characteristics for various values of P, measured, analog simulated and calculated results are shown in Fig. 8, for comparison.

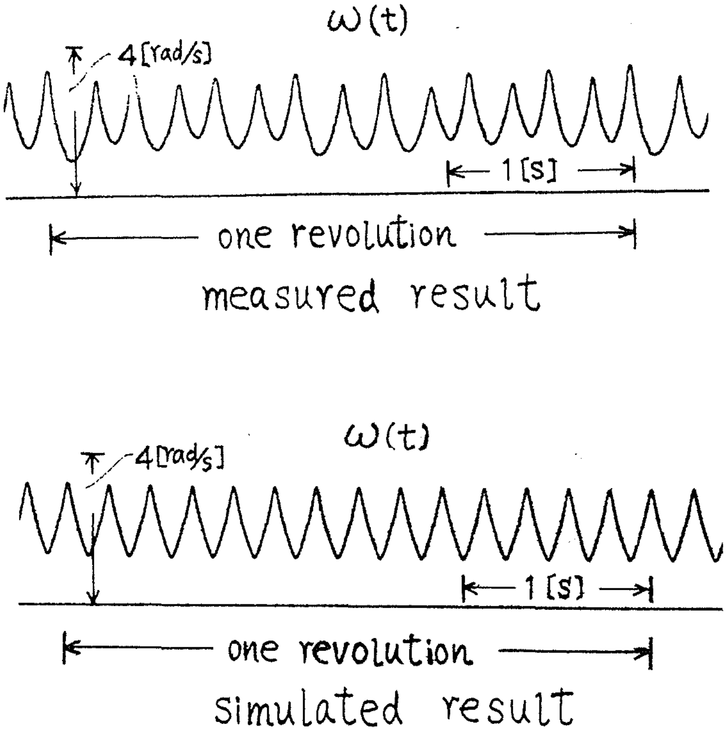

Fig. 6　The shapes of $\omega(t)$, $\bar{\omega}=2$[rad/s], $P=71.2(\Gamma=200)$

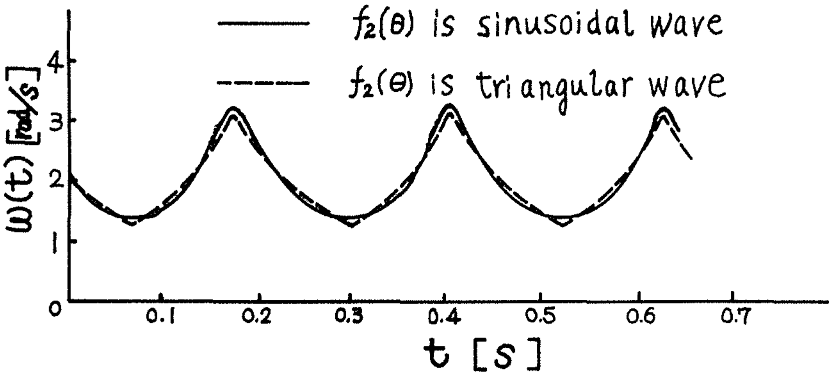

Fig. 7　The shapes of $\omega(t)$, (calculated results), $\bar{\omega}=2$[rad/s], $P=71.2(\Gamma=200)$

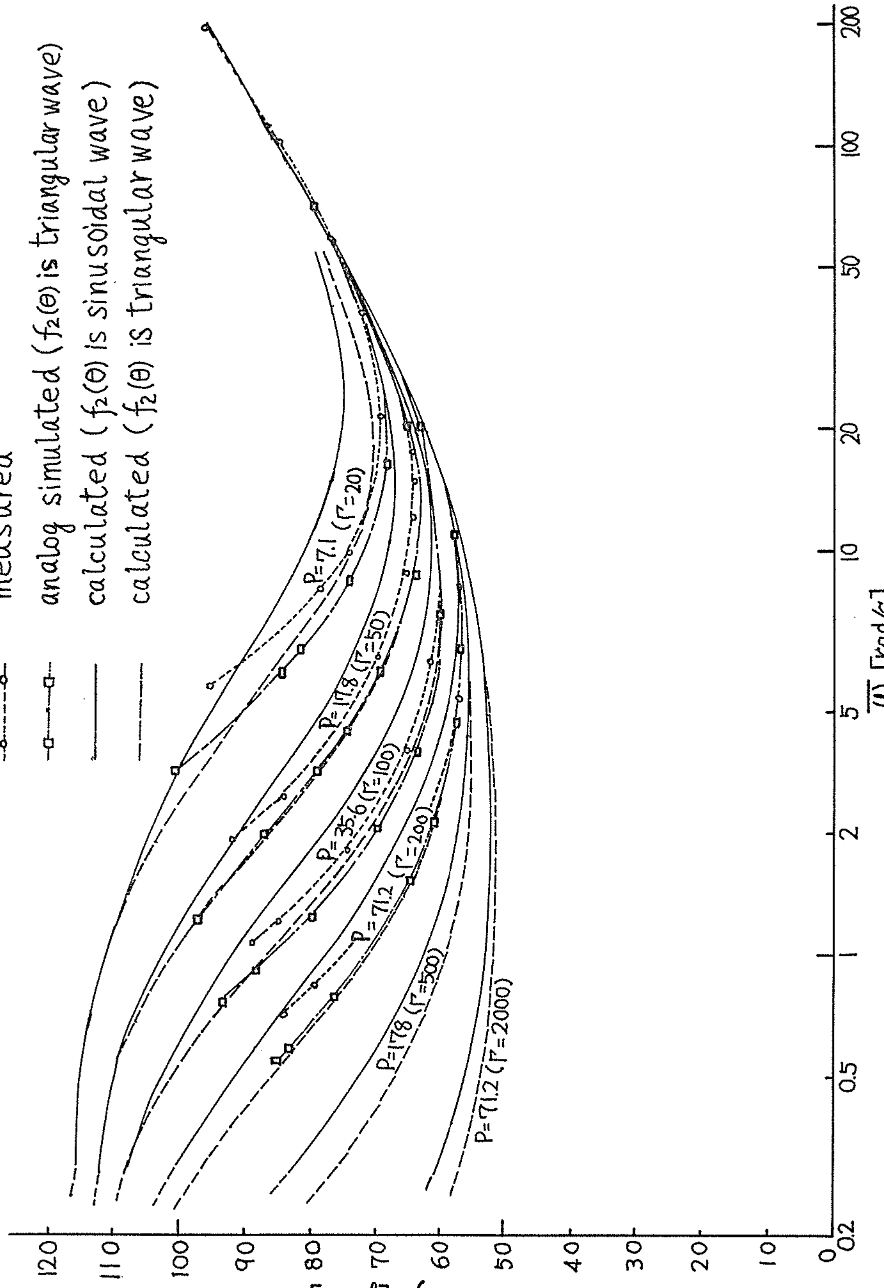

Fig. 8 $\bar{\omega}$-$\bar{f}$ charateristics, (Comparison of measured, analog simulated and calculated results)

The shapes of $\alpha(t)$ obtained from calculations for $\bar{\omega}=2$ rad/s and P=71.2 are shown in Fig. 9, and the relations between $\bar{\omega}$ and α_P obtained from calculations are shown in Fig. 10.

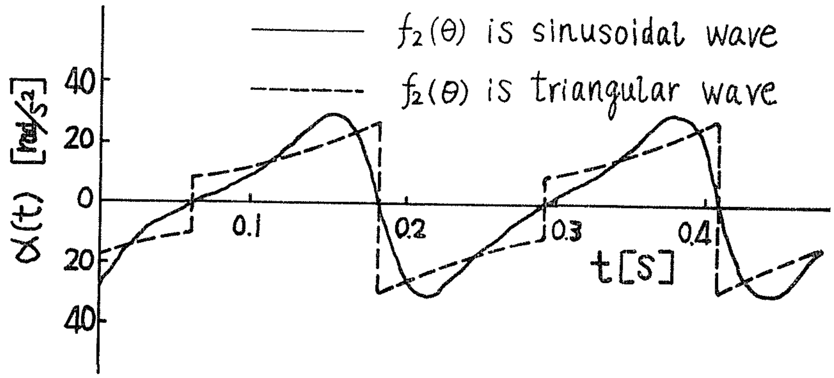

Fig. 9　The shapes of $\alpha(t)$, (calculated results), $\bar{\omega}=2$[rad/s], P=71.2(Γ=200)

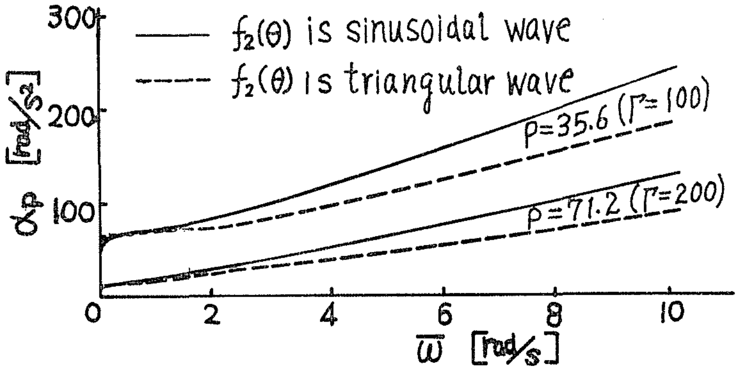

Fig. 10　Relation between $\bar{\omega}$ and α_p

3.2　Discussion

1)　In comparison of the shapes of $\omega(t)$, it is seen from Fig. 6 and Fig. 7 that fairly good results can be obtained if $f_2(\theta)$ is regarded as the sinusoidal wave or the triangular wave. However, since the shapes of the varying parts of the measured friction torque, are not completely uniform, the measured result is slightly different from both the simulated and calculated results.

2) The simulated result in Fig. 6, in which $f_2(\theta)$ is assumed to be the triangular wave and the acceleration torque is included, agrees closely with the calculated result in Fig. 7 in which $f_2(\theta)$ is assumed to be the triangular wave. It seems that this means the effect of the acceleration torque term being almost negligible for such magnitude of $\bar{\omega}$ and P in Fig. 6 and Fig. 7.

3) With respect to the $\bar{\omega}$ - $\bar{f}$ caracteristics in Fig. 8, the simulated results agree very well in most part with the calculated results if $f_2(\theta)$ is the triangular wave, for P>17.8. It appears that this also is due to the same reason as in 2) that the acceleration torque is very small in this range of $\bar{\omega}$ and P.

4) Each $\bar{\omega}$ which is obtained from measurment and analog simulation has a lower limit, however calculated one can take any small value because of neglecting the acceleration torque term in analysis.

5) In the range of small angular velocity of the motor, which has the friction torque varying with angular position, as shown in Fig. 8, $\bar{f}$ decreases as $\bar{\omega}$ increases. Namely a negative slope characteristic is obtained, and even if $\bar{\omega}$ has the same value, $\bar{f}$ takes different values for different values for P. In this analysis, Eqs. (20) and (28) showing $\bar{\omega}$ - $\bar{f}$ cfaracteristics can express these phenomena quantitatively, while in the past these have been explained qualitatively.(1) Therefore these equations appear to be very useful and important.

6) Let α_{PO} and α_{PJ} be the maximum values of angular acceleration when the moment of inertia of the motor is either zero or non-zero respectively, then in the stable region of motor behavior, the following relation exists in general,

$$\alpha_{PO} > \alpha_{PJ}$$

Thus,

$$J_m \alpha_{PO} > J_m \alpha_{PJ}$$

The moment of inertia of the motor used in the study is

$$J_m = 0.0276 \text{ [g-cm-s}^2\text{]}$$

For example, at P=71.2 (Γ=200) and $\bar{\omega}$=2 rad/s, the maximum value of the angular acceleration of the motor is obtained from Fig. 9 or Fig. 10 as follows,

$$\alpha_P \fallingdotseq 30 \text{ [rad/s}^2\text{]}$$

then, the maximum value of the acceleration torque of the motor is

$$J_m \alpha_{PJ} < 30 \times 0.0276 \text{ [g-cm]}$$

This value is smaller than 1.2 per cent of the amplitude of $f_2(\theta)$: (F_P=68 g-cm), therefore it seems that the effect of the acceleration torque is sufficiently negligible under these conditions.

7) In investigation of the system which contains the friction torque varying with angular position : $f_2(\theta)$, it is necessary to know $f_2(\theta)$, and this is identified with f_v(t) which is measured under the completely constant angular velocity, where f_v(t) is the varying part of the friction torque measured with respect

to time. Although the angular velocity may not be constant completely, if the variation of the angular velocity is suppressed in a permitted limit, then $f_v(t)$ can be regarded as $f_2(\theta)$ with a required accuracy. Thus, in order to investigate such a problem as is mentioned above, the analytical results obtained in this paper are available.

4 Conclusion

In a d-c servomotor which has the friction torque varying with angular position, if sufficiently large rate feedback is applied to the motor, it is clarified that the characteristics of the motor in the range of a small angular velocity can be investigated analytically without considering the effect of acceleration torque.

This means that the measuring apparatus can be simplified because detection of an angular accleration is not necessary, for instance, in measurement of the friction torque of the motor to examine the effect of skew, whose armature has skewed slots.

Moreover, as mentioned in Discussion-7), the analytical results obtained in this paper can be also used to investigate the permitted limit of the variation of the angular velocity in which the friction torque varying with angular position and the varying part of the friction torque measured with respect to time can be identified with a required accuracy.

(Most part of this study was presented at the 35th Research Meeting of the Tohoku Branch of SICE in July, 1971.)

Refferences

(1) M. Hattori R. Takanashi and T. Anno : "On the Friction Torque of D-C Servomotor" Trans. SICE, Vol 6, No. 2, pp. 104-108, 1970.

(2) M. Hattori, R. Takanashi and T. Anno : "Friction Torque of D-C Servomotor in Lower Speed and Its Simulation", The 27th Research Meeting Note of the Tohoku Branch of SICE, pp. 1-10, December, 1969.

山形大学紀要（工学）第12巻　第1号　昭和47年1月

回転角によって変動する摩擦のある直流サーボモータに大きな速度帰還をかけた場合の特性の解析

服部正行*・高梨良一**・安濃恒友*

（山形大学工学部*・山形大学工業短期大学部**）

直流サーボモータにおいて，回転角によって変動する摩擦トルクを考慮した場合，モータの関係式が二階の非線形微分方程式となるが，その解は一般的に求める事ができないため，解析的な検討は行なわれていなかった。

本論文では，モータへの速度帰還が十分大きい場合にはモータの関係式において，粘性減衰項（一階の項）に比べて慣性トルクの項（二階の項）の影響が非常に小さくなることから，慣性トルクの項を省略してモータの特性の解析を行なった。またこの解析結果を検討するため，一例として，市販の代表的なサーボモータについて特性を計算し，そのモータの実測値ならびにアナログシミュレーションの結果との比較を行なった。

その結果，本論文で示した解析結果は，たとえば，時間平均回転角速度に対する時間平均摩擦トルク特性における負傾斜現象のような，これまで定性的に説明されていた直流サーボモータの速度帰還が大きい場合の特性を定量的に表わし得ることが明らかとなった。

なお，解析の際に省略した慣性トルクの項が特性にどの程度の影響を与えるか等についても考察を行なっている。